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# Spectrally arbitrary patterns

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## Abstract

An $n \times n$ sign pattern matrix $S$ is an inertially arbitrary pattern (IAP) if each nonnegative triple $(n_1, n_2, n_3)$ with $n_1 + n_2 + n_3 = n$ is the inertia of a matrix with sign pattern $S$. Analogously, $S$ is a spectrally arbitrary pattern (SAP) if, for any given real monic polynomial $r(x)$ of order $n$, there is a matrix with sign pattern $S$ and characteristic polynomial $r(x)$. Focusing on tree sign patterns, consider the $n \times n$ tridiagonal sign pattern $T_n$ with each superdiagonal entry positive, each subdiagonal entry negative, the $(1, 1)$ entry negative, the $(n, n)$ entry positive, and every other entry zero. It is conjectured that $T_n$ is an IAP. By constructing matrices $A_n$ with pattern $T_n$, it is proved that $T_n$ allows any inertia with $n_3 \in \{0, 1, 2, n-1, n\}$ for all $n \geqslant 2$. This leads to a proof of the conjecture for $n \leqslant 5$. The truth of the conjecture is extended to $n \leqslant 7$ by showing the stronger result that $T_n$ is a SAP. The proof of this latter statement involves finding a matrix $A_n$ with pattern $T_n$ that is nilpotent. Further questions about patterns that are SAPs and IAPs are considered. © 2000 Elsevier Science Inc. All rights reserved.

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PII: S 0 0 2 4 - 3 7 9 5 ( 0 0 ) 0 0 0 2 6 - 4

## 1. Introduction and general observations

The set of all eigenvalues (counting multiplicities) of a matrix $A$ is denoted by $\sigma(A)$, and the *inertia of matrix* $A$ is the ordered triple $i(A) = (i_+(A), i_-(A), i_0(A))$, in which $i_+(A)$, $i_-(A)$ and $i_0(A)$ are the numbers of elements of $\sigma(A)$ with positive, negative and zero real parts, respectively. An $n \times n$ (sign) *pattern* (matrix) $S = [s_{ij}]$ has $s_{ij} \in \{+, -, 0\}$ and defines a *sign pattern class* of real matrices

$$Q(S) = \{A = [a_{ij}] \in M_n(R) : sign(a_{ij}) = s_{ij} \text{ for all } i, j\}.$$

The *inertia of pattern* $S$ is the set of ordered triples $i(S) = \{i(A) : A \in Q(S)\}$. Matrix $A$ is *stable* if $i(A) = (0, n, 0)$. Pattern $S$ is *sign stable* [5] if $i(S) = \{(0, n, 0)\}$, and *potentially stable* [7] if $(0, n, 0) \in i(S)$.

We are interested in patterns $S$ with $n \geqslant 2$ that allow every inertia, that is $|i(S)| = (n + 1)(n + 2)/2$, the number of possible inertias over all matrices in $M_n(R)$. To study such patterns, we introduce some definitions. Pattern $S$ is an *inertially arbitrary pattern* (IAP) if $(n_1, n_2, n_3) \in i(S)$ for every nonnegative triple $(n_1, n_2, n_3)$ with $n_1 + n_2 + n_3 = n$. More generally, $S$ is a *spectrally arbitrary pattern* (SAP) if given any monic polynomial $r(x)$ of order $n \geqslant 2$ with real coefficients, there exists $A \in Q(S)$ such that the characteristic polynomial of $A$ is $r(x)$. That is, $S$ is a SAP if there exists $A \in Q(S)$ having any possible spectrum of a real matrix, namely any set of $n$ complex numbers with any nonreals occurring as conjugate pairs. Obviously, if $S$ is a SAP, then $S$ is an IAP, but the converse is an interesting open question.

The problems considered in this paper are closely related to inverse eigenvalue and eigenvalue completion problems (see, e.g., [3,4,8]). Over the complex field the diagonal entries of an otherwise fixed matrix can be chosen so that the resultant matrix has an arbitrary (complex) spectrum [3]. However, over the real field, no such general results seem to be known.

We assume throughout that $S$ is irreducible and $A \in Q(S)$ is real. If $S$ is reducible, and each of its irreducible components is a SAP, then $S$ is a SAP, and we believe that the converse is true. We confine our attention mainly to *tree* (sign) *patterns*, i.e., patterns for which the undirected graph of $S$, namely $G(S) = (V, E)$ with $V = \{1, \dots, n\}$ and $E = \{\{i, j\} : s_{ij} \text{ and } s_{ji} \neq 0\}$, is a tree (see, e.g., [7]). Note that if $S$ is a tree pattern, then the diagonal entries can be $+$, $-$, or 0 (that is, we allow loops, namely edges of the form $\{i, i\}$, in $G(S)$). We are especially interested in minimal patterns. Pattern $S$ is a *minimal* IAP (MIAP) if $S$ is an IAP, but is not an IAP if one or more nonzero entries is replaced by zero. Similarly, $S$ is a *minimal* SAP (MSAP) if $S$ is a SAP, but is not a SAP if one or more nonzero entries is replaced by zero. For example, it is easy to check that the $2 \times 2$ pattern

$$T_2 = \begin{bmatrix} - & + \\ - & + \end{bmatrix}$$

is a MSAP (and it is also a MIAP).

Observe that if $S$ is a SAP (IAP) then $-S$ is also a SAP (IAP). Each property is also preserved under transposition, permutation similarity and signature similarity.

Thus SAPs (IAPs) may be identified up to these equivalences. Obvious necessary conditions for $S$ to be an IAP are that $S$ is potentially stable, and that $Q(S)$ allows a positive and a negative principal minor of order $k$ for all $k = 1, \dots, n$. In particular, ($k = 1$) $S$ has at least one positive and one negative diagonal entry; and ($k = n$) $S$ is not a sign nonsingular pattern. In terms of $G(S)$ the first condition states that there must be two loops of opposite sign; and the determinant condition states that there must be two nonzero signed transversals of opposite sign.

When $S$ is a tree pattern, it has $2(n-1)$ off-diagonal nonzero entries; thus a tree MIAP (MSAP) has at least $2n$ nonzero entries. We are interested in the existence of tree MSAPs (MIAPs) with this minimum number of nonzero entries. Identifying patterns up to equivalence, $T_2$ is the only MSAP (MIAP) for $n = 2$. For $n = 3$, if $S$ is a tree pattern, then $S$ is permutation similar to a tridiagonal pattern. The determinant condition gives that the two nonzero diagonal entries for a MSAP (MIAP) must be the $(1, 1)$ and $(3, 3)$ entries. By checking all tridiagonal patterns with this restriction that are potentially stable (see [7], Fig. 2), it can be verified that (up to equivalence)

$$T_3 = \begin{bmatrix} - & + & 0 \\ - & 0 & + \\ 0 & - & + \end{bmatrix}$$

is the only tree $3 \times 3$ MSAP (MIAP) with six nonzero entries. A broader discussion of tree MSAPs is given in Section 6.

## 2. Antipodal tridiagonal pattern $T_n$

Motivated by the $n = 2$ and 3 cases discussed above, we introduce for $n \geqslant 2$ the $n \times n$ antipodal tridiagonal pattern $T_n$ defined as

$$T_n = \begin{bmatrix} - & + & 0 & \cdots & \cdots & 0 \\ - & 0 & + & \ddots & & \vdots \\ 0 & - & 0 & + & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & - & 0 & + \\ 0 & \cdots & \cdots & 0 & - & + \end{bmatrix}. \tag{1}$$

(An antipodal tridiagonal pattern is one in which only the first and last diagonal entries are nonzero.) Since $A = [a_{ij}] \in Q(T_n)$ is tridiagonal, its off-diagonal entries enter into its characteristic equation only as products $a_{i,i+1}a_{i+1,i}$. Thus it suffices to consider only those matrices $A_n \in Q(T_n)$ that have superdiagonal entries normalized to 1, i.e.,

$$A_n = \begin{bmatrix} -a_0 & 1 & 0 & \cdots & \cdots & 0 \\ -a_1 & 0 & 1 & \ddots & & \vdots \\ 0 & -a_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & -a_{n-2} & 0 & 1 \\ 0 & \cdots & \cdots & 0 & -a_{n-1} & a_n \end{bmatrix},$$

in which $a_0, a_1, \dots, a_n$ are positive real numbers.

In Sections 3 and 4, we address the following conjectures. Note that the truth of Conjecture 2 would imply the truth of Conjecture 1, but not conversely.

**Conjecture 1.** $T_n$ is a minimal inertially arbitrary pattern (MIAP).

**Conjecture 2.** $T_n$ is a minimal spectrally arbitrary pattern (MSAP).

The following result, which is relevant to Conjecture 1, shows that if $i(A_n) = (n_1, n_2, n_3)$, then the parameters $a_i$ can be reordered so that the inertia of the resulting matrix has $n_1$ and $n_2$ interchanged.

**Lemma 3.** *If* $i(A_n) = (n_1, n_2, n_3)$, *then*

$$\hat{A}_n = \begin{bmatrix} -a_n & 1 & 0 & \cdots & \cdots & 0 \\ -a_{n-1} & 0 & 1 & \ddots & & \vdots \\ 0 & -a_{n-2} & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & -a_2 & 0 & 1 \\ 0 & \cdots & \cdots & 0 & -a_1 & a_0 \end{bmatrix}$$

*has inertia* $(n_2, n_1, n_3)$.

**Proof.** If $P = [p_{ij}]$ denotes the $n \times n$ reverse permutation matrix, with $p_{i,n-i+1} = 1$, then $i(P(-A_n^{\mathrm{T}})P^{\mathrm{T}}) = (n_2, n_1, n_3)$. Since

$$P(-A_n^{\mathrm{T}})P^{\mathrm{T}} = \begin{bmatrix} -a_n & -1 & 0 & \cdots & \cdots & 0 \\ a_{n-1} & 0 & -1 & \ddots & & \vdots \\ 0 & a_{n-2} & 0 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & a_2 & 0 & -1 \\ 0 & \cdots & \cdots & 0 & a_1 & a_0 \end{bmatrix},$$

and its off-diagonal products are equal to those of $\hat{A}_n$, the matrix $\hat{A}_n$ also has this inertia. $\square$

We conclude this section by defining tridiagonal matrices $B_n$ and $C_n$ that are used in the next section. For $n \geqslant 2$, let

$$B_n = \begin{bmatrix} -1 & 1 & 0 & \cdots & \cdots & 0 \\ -1 & 0 & 1 & \ddots & & \vdots \\ 0 & -1 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & -1 & 0 & 1 \\ 0 & \cdots & \cdots & 0 & -1 & 0 \end{bmatrix},$$

with $B_1 = [-1]$. Note that $i(B_n) = (0, n, 0)$, since the sign pattern associated with $B_n$ is sign stable (see, e.g., [5]). Similarly, for $n \geqslant 2$, let

$$C_n = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ -1 & 0 & 1 & \ddots & & \vdots \\ 0 & -1 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & -1 & 0 & 1 \\ 0 & \cdots & \cdots & 0 & -1 & 1 \end{bmatrix},$$

with $C_1 = [\,1\,]$. From $i(B_n) = (0, n, 0)$, it follows that $i(C_n) = (n, 0, 0)$ by an argument as in Lemma 3. $\square$

## 3. Inertia results for $T_n$ for general $n$

In this section we prove by constructing matrices in $Q(T_n)$ that $(n_1, n_2, n_3) \in i(T_n)$ for $n_3 \in \{0, 1, 2, n-1, n\}$ for all values of $n \geqslant 2$. The proofs for the first three of these cases depend on the fact that the eigenvalues of a matrix are continuous functions of the matrix entries.

**Theorem 4.** *Consider $A_n$, $n \geqslant 2$, and $k \in \{0, 1, \ldots, n\}$. Let $a_k = \epsilon$ and $a_j = 1$ for all $j \neq k$. Then for all sufficiently small $\epsilon > 0$, $i(A_n) = (n-k, k, 0)$.*

**Proof.** For $k = 0$ or $n$ and all sufficiently small $\epsilon > 0$, $A_n$ is a small perturbation of $C_n$ or $B_n$, respectively. Since the eigenvalues of $C_n$ and $B_n$ are in the open right and left half planes, respectively, by continuity the inertia of $A_n$ is the same as that of $C_n$ or $B_n$ for all sufficiently small $\epsilon > 0$, namely $(n-k, k, 0)$. For $k \neq 0$ or $n$, $A_n$ is a small perturbation of a block upper triangular matrix with diagonal blocks $B_k$ and $C_{n-k}$, from which the result similarly follows. □

Note that when $k = n$, Theorem 4 shows that $T_n$ is potentially stable.

**Theorem 5.** *Consider* $A_n, n \geqslant 2$, *and* $k \in \{0, 1, \ldots, n-1\}$. *Let* $a_k = a_{k+1} = \epsilon$ *and all other* $a_j = 1$. *Then for all sufficiently small* $\epsilon > 0$, $i(A_n) = (n-k-1, k, 1)$.

**Proof.** The matrix $A_n$ has exactly two nonzero signed transversals, which are equal to $\epsilon$ and $-\epsilon$, and thus $\det A_n = 0$. Therefore, $A_n$ has at least one eigenvalue equal to 0. For $k = 0$ or $n-1$ and all sufficiently small $\epsilon > 0$, $A_n$ is a small perturbation of a block upper triangular matrix with diagonal blocks $C_{n-1}$ or $B_{n-1}$, respectively, and $[\,0\,]$. Otherwise, $A_n$ is a small perturbation of a block upper triangular matrix with diagonal blocks $B_k$, $[\,0\,]$ and $C_{n-k-1}$. For all $k$, the result now follows by a continuity argument as in the proof of Theorem 4. □

The construction in the following result guarantees that $a_0, a_1, \ldots, a_n$ can be chosen so that $i_0(A_n) = 2$, with two eigenvalues of $A_n$ equal to 0. Let $A_n(k)$ denote the $(n-1) \times (n-1)$ principal submatrix of $A_n$ obtained by deleting row and column $k$ of $A_n$.

**Theorem 6.** *Consider* $A_n$, *where* $n \geqslant 2$, *and* $k \in \{0, 1, \ldots, n-2\}$. *Let all* $a_j = 1$ *except*

(i) $a_k = a_{k+2} = \epsilon,\ \ a_{k+1} = \epsilon^2$ *if n is even*;

(ii) $a_k = \epsilon,\ \ a_{k+1} = \frac{\epsilon^2}{1+\epsilon},\ \ a_{k+2} = \frac{\epsilon}{1+\epsilon}$ *if n is odd and k is even*;

(iii) $a_k = \frac{\epsilon}{1+\epsilon},\ \ a_{k+1} = \frac{\epsilon^2}{1+\epsilon},\ \ a_{k+2} = \epsilon$ *if n is odd and k is odd.*

*Then for all sufficiently small* $\epsilon > 0$, $i(A_n) = (n-k-2, k, 2)$.

**Proof.** If $n = 2$, then $k = 0$ and the characteristic polynomial of $A_n$ is $z^2$, so $i(A_n) = (0, 0, 2)$ as claimed. We now assume that $n \geqslant 3$, and first restrict consideration to cases (i) and (ii).

The matrix $A_n$ has exactly two nonzero signed transversals, equal to $\epsilon^2$ and $-\epsilon^2$ in (i) and equal to $\frac{\epsilon^2}{1+\epsilon}$ and $\frac{-\epsilon^2}{1+\epsilon}$ in (ii), so that $\det A_n = 0$. Furthermore, in case (i),

$$\det A_n(k+1) = (-1)^k \epsilon = -\det A_n(k+2),$$

$$\det A_n(j) = (-1)^{j+1} \epsilon^2 \text{ if } j \neq k+1, k+2;$$

whereas in case (ii),

$$\det A_n(k+1) = \frac{\epsilon}{1+\epsilon}, \quad \det A_n(k+2) = -\epsilon, \quad \det A_n(k+3) = \frac{\epsilon^2}{1+\epsilon},$$

$$\det A_n(j) = (-1)^{j+1} \frac{\epsilon^2}{1+\epsilon} \quad \text{for } 1 \leqslant j \leqslant k \text{ and } k+4 \leqslant j \leqslant n.$$

(These minors can be obtained by determining all nonzero signed transversals in $G(T_n)$ with the appropriate vertex and its adjacent edges removed.) In both cases

$$\sum_{j=1}^{n} \det A_n(j) = 0,$$

and consequently the constant and linear terms in the characteristic polynomial of $A_n$ are 0. Thus, $A_n$ has at least two eigenvalues equal to 0.

In the remainder of this proof, we denote $A_n$ by $A_n(\epsilon)$ (to emphasize its dependence on the parameter $\epsilon$). For all $k$, $i(A_n(0)) = (n-k-2, k, 2)$ since $A_n(0)$ is a block upper triangular matrix with diagonal blocks

$$B_k, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } C_{n-k-2}$$

(where if $k=0$ or $n-2$, then $B_k$ or $C_{n-k-2}$, respectively, does not appear). Since $A_n(\epsilon)$ has at least two eigenvalues equal to 0, by continuity $i(A_n(\epsilon)) = (n-k-2, k, 2)$ for all sufficiently small $\epsilon > 0$, completing the proof of cases (i) and (ii).

Case (iii) follows by applying Lemma 3 to case $(ii)$ and noting that if $0 \leqslant k \leqslant n-3$ with $k$ even, then $1 \leqslant n-k-2 \leqslant n-2$ and $n-k-2$ is odd. □

Unlike the continuity arguments used above, the proof of the following theorem depends on a relationship between the characteristic polynomials of $A_n$ and a particular tridiagonal skew symmetric matrix.

**Theorem 7.** *Consider $A_n$, where $n \geqslant 2$, with $a_n = 1$ and $a_i = \epsilon$ for $0 \leqslant i \leqslant n-1$. Then*

$$i(A_n) = \begin{cases} (0,0,n) & \text{if } \epsilon = 1, \\ (1,0,n-1) & \text{if } \epsilon < 1, \\ (0,1,n-1) & \text{if } \epsilon > 1. \end{cases}$$

**Proof.** For $n \geqslant 2$, define $V_n = [v_{ij}]$ to be the $n \times n$ tridiagonal skew-symmetric matrix with $v_{i,i+1} = \sqrt{\epsilon} = -v_{i+1,i}$ for $1 \leqslant i \leqslant n-1$, and $v_{ij} = 0$ otherwise; define $V_1 = [\,0\,]$. For $n \geqslant 2$, define the $n \times n$ matrix $U_n = [u_{ij}]$ so that $U_n = V_n$, except that $u_{11} = -\epsilon$ and $u_{nn} = 1$. Letting $p_n(z) = \det(zI - A_n)$ and noting that $p_n(z) = \det(zI - U_n)$, expansion of the latter determinant gives

$$\begin{aligned} p_n(z) = {} & \det(zI - V_n) - \det(zI - V_{n-1}) + \epsilon \det(zI - V_{n-1}) \\ & - \epsilon \det(zI - V_{n-2}), \end{aligned}$$

where $\det(zI - V_0) = 1$. Letting $\phi_n(z) = \det(zI - V_n)$ and $\phi_0(z) = 1$ gives

$$p_n(z) = \phi_n(z) - (1-\epsilon)\phi_{n-1}(z) - \epsilon\, \phi_{n-2}(z).$$

Since the characteristic polynomial of $V_n$ satisfies the recurrence relation

$$\phi_n(z) = z\phi_{n-1}(z) + \epsilon\, \phi_{n-2}(z)$$

(see [1, p. 55]), the above reduces to

$$p_n(z) = (z - 1 + \epsilon)\phi_{n-1}(z).$$

Thus, the eigenvalues of $A_n$ are $1 - \epsilon$, together with the eigenvalues of $V_{n-1}$, which are all pure imaginary. The results follow from the sign of $1 - \epsilon$. $\square$

Note that the eigenvalues of $V_{n-1}$ can be shown to be $2i\sqrt{\epsilon}\cos\frac{k\pi}{n}$, for $1 \leqslant k \leqslant n-1$. Thus, $V_{n-1}$ has one eigenvalue equal to 0 if $n$ is even and none if $n$ is odd.

Experimentation with various numerical realizations of $A_n$ has led to the following conjecture.

**Conjecture 8.** Consider $A_n$, with $n \geqslant 2$, and $k \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. If $a_i = 1$ for $i < k$ or $i > n - k$ and $a_i = \epsilon$ otherwise, then $i(A_n) = (k, k, n - 2k)$ for all sufficiently small $\epsilon > 0$.

Clearly the conjecture is true for $n$ even and $k = \frac{n}{2}$ by Theorem 4, and for $n$ odd and $k = (n-1)/2$ by Theorem 5. The conjecture has also been verified for all possible $k$ using $\epsilon = 0.01$ and $2 \leqslant n \leqslant 7$.

Theorems 4 through 7 show that $T_n$ is a MIAP for $2 \leqslant n \leqslant 4$, and that $i(T_5)$ contains all possible triples except possibly (1,1,3), (2,0,3) and (0,2,3). The inertia (1,1,3) can be achieved with $a_i$ as in Conjecture 8. The inertia (2,0,3) can be achieved by

$$(a_0, a_1, \ldots, a_5) = \left(\frac{\sqrt{2}}{2}, 1 - \frac{\sqrt{2}}{2}, \sqrt{2} - 1, 1, 1, 1\right),$$

which yields the characteristic polynomial

$$z^5 + \left(\frac{\sqrt{2}}{2} - 1\right) z^4 + 2z^3.$$

By using Lemma 3 and the fact that (2,0,3) can be achieved, it follows that (0,2,3) can also be achieved. Hence we have constructed matrices to prove that Conjecture 1 is true for $2 \leqslant n \leqslant 5$.

## 4. Spectral results for $T_n$ for small $n$

Another approach to showing that $T_n$ is a MIAP is to show that $T_n$ is a MSAP. We show below that this is true for $2 \leqslant n \leqslant 7$, hence Conjecture 2 (and thus Conjecture 1) is true for $2 \leqslant n \leqslant 7$.

**Theorem 9.** *For $2 \leqslant n \leqslant 7$ and any real numbers $r_0, r_1, \ldots, r_{n-1}$, there exists $A_n \in Q(T_n)$ such that*

$$\det(zI - A_n) = z^n + r_{n-1}z^{n-1} + r_{n-2}z^{n-2} + \cdots + r_1 z + r_0.$$

**Proof.** Since $A_n \in Q(T_n)$ if and only if $cA_n \in Q(T_n)$ for $c > 0$, and since

$$\det(zI - cA_n) = z^n + cr_{n-1}z^{n-1} + c^2 r_{n-2}z^{n-2} + \cdots + c^n r_0,$$

it suffices to show that the theorem is valid for $(r_0, \ldots, r_{n-1})$ arbitrarily close to $(0, \ldots, 0)$. We first consider the case $n = 4$, then discuss the remaining values of $n$.

The characteristic polynomial of $A_4$ is

$$z^4 + (a_0 - a_4)z^3 + (a_1 + a_2 + a_3 - a_0a_4)z^2 + (a_0a_2 + a_0a_3 - a_1a_4 - a_2a_4)z \\ + a_1a_3 - a_0a_2a_4.$$

We seek positive real numbers $a_0, a_1, \ldots, a_4$ such that

$$\begin{aligned} &a_0 - a_4 - r_3 = 0, \\ &a_1 + a_2 + a_3 - a_0a_4 - r_2 = 0, \\ &a_0a_2 + a_0a_3 - a_1a_4 - a_2a_4 - r_1 = 0, \\ &a_1a_3 - a_0a_2a_4 - r_0 = 0. \end{aligned} \tag{2}$$

If $A_4$ is nilpotent, namely $r_0 = r_1 = r_2 = r_3 = 0$, it is easy to verify that a solution to (2) is

$$\hat{a}_0 = \hat{a}_4 = 1, \quad \hat{a}_1 = \hat{a}_3 = \sqrt{2} - 1, \quad \hat{a}_2 = 3 - 2\sqrt{2}.$$

Setting $a_4 = \hat{a}_4 = 1$, define functions $f_1, f_2, f_3, f_4$ of $a_0, a_1, a_2, a_3, r_0, r_1, r_2, r_3$ to be the left sides of the equations in (2), in order. Note that the functions $f_i$ have continuous partial derivatives with respect to all eight variables. Further,

$$\frac{\partial(f_1, f_2, f_3, f_4)}{\partial(a_0, a_1, a_2, a_3)} = \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ a_2 + a_3 & -1 & a_0 - 1 & a_0 \\ -a_2 & a_3 & -a_0 & a_1 \end{bmatrix} = a_0^2 + a_0a_1 + a_0 + a_3.$$

Evaluating the Jacobian at $(\hat{a}_0, \hat{a}_1, \hat{a}_2, \hat{a}_3, 0, 0, 0, 0)$ yields

$$J_4 = \hat{a}_0^2 + \hat{a}_0\hat{a}_1 + \hat{a}_0 + \hat{a}_3 = 2\sqrt{2}, \tag{3}$$

which is nonzero. By the Implicit Function Theorem, for $(r_0, r_1, r_2, r_3)$ sufficiently close to $(0, 0, 0, 0)$, there are unique continuous functions $a_0, a_1, a_2, a_3$ of $r_0, r_1, r_2, r_3$ that maintain $f_1 = f_2 = f_3 = f_4 = 0$. Since $(\hat{a}_0, \hat{a}_1, \hat{a}_2, \hat{a}_3)$ is a positive vector, $(a_0, a_1, a_2, a_3)$ is a positive vector for $(r_0, r_1, r_2, r_3)$ sufficiently close to $(0, 0, 0, 0)$, thereby yielding a realization of $A_4$ with the desired characteristic polynomial.

The above proof for the case $n = 4$ relies on two facts: first, $A_4$ has a nilpotent realization and second, a certain Jacobian is nonzero. Extending the proof for $A_4$ to $A_2$ through $A_7$ requires, first, the existence of such nilpotent realizations, so we list them here (where approximations are given for $n = 7$).

| $n$ | $\hat{a}_0$ | $\hat{a}_1$ | $\hat{a}_2$ | $\hat{a}_3$ | $\hat{a}_4$ | $\hat{a}_5$ | $\hat{a}_6$ | $\hat{a}_7$ |
|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 1 | 1 | | | | | |
| 3 | 1 | 1/2 | 1/2 | 1 | | | | |
| 4 | 1 | $\sqrt{2}-1$ | $3-2\sqrt{2}$ | $\sqrt{2}-1$ | 1 | | | |
| 5 | 1 | $\frac{3-\sqrt{5}}{2}$ | $\frac{\sqrt{5}-2}{2}$ | $\frac{\sqrt{5}-2}{2}$ | $\frac{3-\sqrt{5}}{2}$ | 1 | | |
| 6 | 1 | $\frac{\sqrt{3}-1}{2}$ | $\frac{3\sqrt{3}-5}{2}$ | $7-4\sqrt{3}$ | $\frac{3\sqrt{3}-5}{2}$ | $\frac{\sqrt{3}-1}{2}$ | 1 | |
| 7 | 1 | 0.357 | 0.088 | 0.055 | 0.055 | 0.088 | 0.357 | 1 |

Second, for $2 \leqslant n \leqslant 7$, it must be shown that a certain Jacobian $J_n$ is nonzero. Note that calculating $J_n$ can be simplified by deleting $r_0, \ldots, r_{n-1}$ from $f_i$ before differentiating without changing the result. Setting $a_n = \hat{a}_n = 1$, $\partial(f_1, \ldots, f_n)/\partial(a_0, \ldots, a_{n-1})$ must be nonzero at $(a_0, \ldots, a_{n-1}, r_0, \ldots, r_{n-1}) = (\hat{a}_0, \ldots, \hat{a}_{n-1}, 0, \ldots, 0)$, where $f_i$ denotes the coefficient of $z^{n-i}$ in $\det(zI - A_n)$. Using the symmetry apparent in the table above (see Theorem 11 below), the Jacobians may be written as follows:

$$J_2 = 1, \quad J_3 = 2, \quad J_4 = 2(\hat{a}_1 + 1), \quad J_5 = 4\hat{a}_1(\hat{a}_2 + 1),$$

$$J_6 = 4\hat{a}_1[(\hat{a}_1 - \hat{a}_2)^2 + \hat{a}_2(\hat{a}_1 + 1)(\hat{a}_3 + 1)],$$

$$J_7 = 8\hat{a}_1^2\hat{a}_2[(\hat{a}_1 - \hat{a}_3)^2 + \hat{a}_3(\hat{a}_2 + 1)(\hat{a}_3 + 1)].$$

Since $J_2, J_3, \ldots, J_7$ are all positive, they are nonzero, as required. Thus the proof in the case $n = 4$ can be extended to $2 \leqslant n \leqslant 7$. □

Since $a_i$ can be chosen so that $A_2, A_3, \ldots, A_7$ can have any possible characteristic polynomial and $A_n$ has exactly $2n$ nonzero entries, $T_n$ is a MSAP, and hence a MIAP for $2 \leqslant n \leqslant 7$. Extending the proof of Theorem 9 to a larger value of $n$ requires finding a nilpotent realization of $A_n \in Q(T_n)$ and showing that $J_n \neq 0$, both of which become prohibitively difficult as $n$ increases. This method can also be used with patterns other than $T_n$.

**Observation 10.** In order to use the method from the proof of Theorem 9 to show that an $n \times n$ irreducible pattern $Y$ is a SAP, consider $X \in Q(Y)$ with its nonzero entries specified by positive parameters $x_0, \ldots, x_k$. If $X$ has a nilpotent realization $\hat{X}$ with $(x_0, \ldots, x_k) = (\hat{x}_0, \ldots, \hat{x}_k)$, and there are $n$ $x_i$'s, say $x_0, \ldots, x_{n-1}$, such that the Jacobian of the coefficients of $z^{n-1}, z^{n-2}, \ldots, z^1, z^0$ in $\det(zI - X)$ with respect to $x_0, \ldots, x_{n-1}$ is nonzero when evaluated at $(\hat{x}_0, \ldots, \hat{x}_k)$, then $Y$ is a SAP.

If a nilpotent realization of $A_n$ exists, then the symmetry observed in the table in the proof of Theorem 9 holds for all $n$, as the following result shows.

**Theorem 11.** *If there exists $A_n \in Q(T_n)$ such that* $\det(zI_n - A_n) = z^n$, *then $a_i = a_{n-i}$ for $i = 0, 1, \ldots, \lfloor \frac{n-1}{2} \rfloor$.*

**Proof.** Let $p_n(z) = \det(zI_n - A_n)$ and assume $p_n(z) = z^n$. Since

$$p_2(z) = z^2 + (a_0 - a_2)z + a_1 - a_0a_2 = z^2,$$

it follows that $a_0 = a_2$. Thus the theorem is true for $n = 2$. We now show by induction that the theorem is true for any $n \geqslant 3$. The coefficient of $z^{n-1}$ in $p_n(z)$ is $a_0 - a_n$, so $a_0 = a_n$. The coefficient of $z^{n-3}$ in $p_n(z)$ is

$$a_0a_2 + a_0a_3 + \cdots + a_0a_{n-1} - a_1a_n - a_2a_n - \cdots - a_{n-2}a_n = a_0a_{n-1} - a_1a_n,$$

so $a_1 = a_{n-1}$. Thus $a_i = a_{n-i}$ for $i = 0$ and 1. For $0 \leqslant k \leqslant \lfloor\frac{n-3}{2}\rfloor$, we suppose that $a_i = a_{n-i}$ for $i = 0, \ldots, k$ and show that $a_{k+1} = a_{n-k-1}$, thus completing the induction.

Consider the coefficient of $z^{n-3-2k}$ in $p_n(z)$, which is

$$\sum a_0a_{i_1}a_{i_2}\cdots a_{i_{k+1}} - \sum a_na_{j_1}a_{j_2}\cdots a_{j_{k+1}} = 0,$$

where the first sum is over all $i_t$ such that

$$2 \leqslant i_1 < i_2 < i_3 < \cdots < i_{k+1} \leqslant n-1,$$

$$i_p \geqslant i_{p-1} + 2 \quad \text{for } 2 \leqslant p \leqslant k+1,$$

and the second sum is over all $j_t$ such that

$$1 \leqslant j_1 < j_2 < j_3 < \cdots < j_{k+1} \leqslant n-2,$$

$$j_p \geqslant j_{p-1} + 2 \quad \text{for } 2 \leqslant p \leqslant k+1.$$

Clearly the first sum has the same number of summands as the second sum. We now show that all but one of the summands in the first sum have a corresponding equal summand in the second sum and thus cancel.

Let $y$ be a summand in the first sum that does not have at least one factor belonging to each of the following $k+2$ sets:

$$\{a_0, a_n\}, \{a_1, a_{n-1}\}, \{a_2, a_{n-2}\}, \ldots, \{a_{k+1}, a_{n-k-1}\}.$$

There is a smallest $j \leqslant k+1$ for which neither $a_j$ nor $a_{n-j}$ is a factor of $y$. Construct $\tilde{y}$ from $y$ as follows. Leave unchanged those factors $a_i$ in $y$ for which $j < i < n-j$; replace all other factors $a_i$ in $y$ by $a_{n-i}$. Clearly, $a_n$ is a factor of $\tilde{y}$. Since the factors $a_j$ and $a_{n-j}$ are missing from $y$, and $y$ has no two factors with consecutive subscripts, neither does $\tilde{y}$ and hence $\tilde{y}$ is a summand of the second sum. Moreover, $y = \tilde{y}$ since, by the induction hypothesis, $a_i = a_{n-i}$ for each factor $a_i$ in $y$ that is replaced by $a_{n-i}$. In this way each such summand $y$ in the first sum can be paired with a cancelling summand $\tilde{y}$ in the second sum. Furthermore, this pairing is one-to-one since the construction of $\tilde{y}$ is easily inverted, producing $y$ from $\tilde{y}$. An example of this construction immediately follows the proof.

We now consider the remaining summands. Let $x$ be a summand of the first sum that has at least one factor belonging to each of the above $k+2$ sets. Since $x$ has

$k+2$ factors, and no two of these factors can have consecutive integer subscripts, $x$ is unique and is given by

$$x = \begin{cases} a_0a_2a_4\cdots a_ka_{n-k-1}a_{n-k+1}\cdots a_{n-3}a_{n-1} & \text{when } k \text{ is even,} \\ a_0a_2a_4\cdots a_{k+1}a_{n-k}a_{n-k+2}\cdots a_{n-3}a_{n-1} & \text{when } k \text{ is odd.} \end{cases}$$

Similarly, there is a unique summand in the second sum that has exactly one factor in each of the above $k+2$ sets and is given by

$$\tilde{x} = \begin{cases} a_1a_3a_5\cdots a_{k+1}a_{n-k}a_{n-k+2}\cdots a_{n-2}a_n & \text{when } k \text{ is even,} \\ a_1a_3a_5\cdots a_ka_{n-k-1}a_{n-k+1}\cdots a_{n-2}a_n & \text{when } k \text{ is odd.} \end{cases}$$

Since all summands other than $x$ and $\tilde{x}$ cancel, $x - \tilde{x} = 0$, which implies by the induction hypothesis that $a_{k+1} = a_{n-k-1}$. □

**Example 12.** As an example of the above construction of $\tilde{y}$, consider $n = 20$, $k = 6$ and $y = a_0a_2a_4a_7a_{11}a_{14}a_{16}a_{19}$. For this $y$, $j = 3$ since $\{a_3, a_{17}\}$ is the first set of missing factors and hence $\tilde{y} = a_{20}a_{18}a_4a_7a_{11}a_{14}a_{16}a_1$. Notice that if each $a_i$ in $y$ is replaced by $a_{n-i}$ whenever it is known by the inductive hypothesis that $a_i = a_{n-i}$, then the result $a_{20}a_{18}a_{16}a_7a_{11}a_6a_4a_1$ is not a summand of the second sum, since $a_6$ and $a_7$ have consecutive subscripts.

The above theorem shows that if $A_n$ is nilpotent, then $a_i = a_{n-i}$ for $0 \leqslant i \leqslant \lfloor \frac{n-1}{2} \rfloor$. However, as illustrated by the case with all $a_i = 1$ in Theorem 7, if $i(A_n) = (0, 0, n)$ and $a_i = a_{n-i}$, then $A_n$ is not necessarily nilpotent. If in matrix $A_n$ the restriction that $a_i > 0$ is replaced by $a_i \neq 0$, then the conclusion of Theorem 11 (and its proof) remains valid.

Because questions in this section involving $A_n$ require computation of its characteristic polynomial $p_n(z) = \det(zI - A_n)$, we end the section by displaying the following useful recurrence relation.

**Theorem 13.** *Let* $w_0(z) = 1$, $w_1(z) = z + a_0$, $w_n(z) = zw_{n-1}(z) + a_{n-1}w_{n-2}(z)$ *for* $n \geqslant 2$. *Then* $p_n(z) = w_n(z) - a_nw_{n-1}(z)$ *for* $n \geqslant 2$.

**Proof.** For $n \geqslant 2$, define the matrix $W_n = A_n$, except that the $(n, n)$ entry of $W_n$ is 0, and define $w_n(z) = \det(zI - W_n)$. By expanding $\det(zI - W_n)$ on its last row, it can be seen that $w_n$ satisfies the recurrence relation given in the first sentence of the theorem. Then, expanding $\det(zI - A_n)$ on the entries in its last row yields

$$p_n(z) = (z - a_n)w_{n-1}(z) + a_{n-1}w_{n-2}(z) = w_n(z) - a_nw_{n-1}(z),$$

as required. □

We note that if $P$ is the $n \times n$ reverse permutation matrix, then the matrix $PW_n^{\mathrm{T}}P^{\mathrm{T}}$ (that also has the characteristic polynomial $w_n(z)$) is in Schwarz form (see [1], p. 52), and has inertia $(0, n, 0)$. In fact the sign pattern associated with $W_n$ is sign stable, whereas, if Conjecture 1 is true, then the entries of $A_n$ (that differ from $W_n$ only in the $(n,n)$ entry) can be chosen to give any inertia $(n_1, n_2, n_3)$.

## 5. Spectra of superpatterns of $T_n$

Let $\tilde{T}_n = [\tilde{t}_{ij}]$ be a sign pattern such that $\tilde{t}_{ij} = t_{ij}$ whenever $t_{ij} \neq 0$, and $\tilde{t}_{ij}$ is one of $+$, $-$ or 0 otherwise, where $T_n = [t_{ij}]$ as in (1). Such a pattern $\tilde{T}_n$ is a *superpattern* of $T_n$. For $A_n \in Q(T_n)$, if $a_i$ are such that $i(A_n) = (k, n-k, 0)$, then, by continuity, any sufficiently small perturbation of this $A_n$ has the same inertia. Such a continuity argument is inconclusive if $i_0(A_n) > 0$. We show below for $3 \leqslant n \leqslant 7$ that if the $a_i$ are such that $A_n$ has a certain inertia, then there exists a perturbation, namely $\tilde{A}_n \in Q(\tilde{T}_n)$, with this same inertia. Thus, by the results of Section 4, $\tilde{T}_n$ is an IAP for these values of $n$. In fact, we show the stronger result that, for $3 \leqslant n \leqslant 7$, $\tilde{T}_n$ is a SAP.

**Theorem 14.** *Let $3 \leqslant n \leqslant 7$. If $\tilde{T}_n$ is a fixed superpattern of $T_n$, then $\tilde{T}_n$ is a SAP.*

**Proof.** Let $r(z) = z^n + \sum_{i=0}^{n-1} r_i z^i$, with $r_0, \ldots, r_{n-1}$ any real numbers. As in the proof of Theorem 9, it suffices to show that this theorem is valid for $(r_0, \ldots, r_{n-1})$ arbitrarily close to $(0, \ldots, 0)$.

We first consider the case $n = 4$ and show that there exist positive $a_0, \ldots, a_4$ and appropriately signed $(+, -, 0)$ real numbers $c_1, \ldots, c_8$ so that

$$\begin{bmatrix} -a_0 & 1 & c_1 & c_2 \\ -a_1 & c_3 & 1 & c_4 \\ c_5 & -a_2 & c_6 & 1 \\ c_7 & c_8 & -a_3 & a_4 \end{bmatrix}$$

is a numerical realization of $\tilde{A}_4 \in Q(\tilde{T}_4)$ and its characteristic polynomial, denoted by $p(z; a_0, \ldots, a_4, c_1, \ldots, c_8)$, satisfies

$$p(z; a_0, \ldots, a_4, c_1, \ldots, c_8) - r(z) = 0. \tag{4}$$

By Theorem 9, there are positive real numbers $b_0, b_1, \ldots, b_4$ such that

$$p(z; b_0, b_1, b_2, b_3, b_4, 0, \ldots, 0) - r(z) = 0.$$

Moreover, by taking $(r_0, \ldots, r_3)$ sufficiently close to $(0, \ldots, 0)$ and using the continuity argument from the proof of Theorem 9, $(b_0, \ldots, b_4)$ can be made arbitrarily close to $(\hat{a}_0, \ldots, \hat{a}_4)$, with $b_4 = \hat{a}_4 = 1$. We seek $a_0, \ldots, a_4$ positive and $c_1, \ldots, c_8$ that satisfy (4), equivalently,

$$\begin{aligned} &a_0 - a_4 - h_1(a_0, \ldots, a_4, c_1, \ldots, c_8) - r_3 = 0, \\ &a_1 + a_2 + a_3 - a_0 a_4 - h_2(a_0, \ldots, a_4, c_1, \ldots, c_8) - r_2 = 0, \\ &a_0 a_2 + a_0 a_3 - a_1 a_4 - a_2 a_4 - h_3(a_0, \ldots, a_4, c_1, \ldots, c_8) - r_1 = 0, \\ &a_1 a_3 - a_0 a_2 a_4 - h_4(a_0, \ldots, a_4, c_1, \ldots, c_8) - r_0 = 0, \end{aligned} \tag{5}$$

in which $h_1, \ldots, h_4$ are polynomial functions each of whose terms contain at least one $c_i$ as a factor. Setting $a_4 = b_4 = 1$, define the functions $g_1, \ldots, g_4$ of $(a_0, \ldots, a_3, c_1, \ldots, c_8)$ to be the left sides of the equations in (5), in order. Observe that $g_1, \ldots, g_4$ have continuous partial derivatives with respect to all twelve variables and that

$$g_i(b_0, \ldots, b_3, 0, \ldots, 0) = 0, \; i = 1, \ldots, 4.$$

Let $\tilde{J}_4$ be $\partial(g_1, \ldots, g_4)/\partial(a_0, \ldots, a_3)$ evaluated at $(b_0, \ldots, b_3, 0, \ldots, 0)$. Calculating $\tilde{J}_4$ can be simplified by deleting $r_0, \ldots, r_3$ from $g_i$, and by setting $(c_1, \ldots, c_8) = (0, \ldots, 0)$ before taking any partial derivatives since terms involving one or more $c_i$ do not influence the value of the Jacobian. Thus, the functions $g_1, \ldots, g_4$ can be replaced by $f_1, \ldots, f_4$ from the proof of Theorem 9, yielding

$$\tilde{J}_4 = b_0^2 + b_0 b_1 + b_0 + b_3,$$

which is the same Jacobian given by (3) with $\hat{a}_i$ replaced by $b_i$. Since $J_4 \neq 0$ and $(b_0, b_1, b_2, b_3)$ can be made arbitrarily close to $(\hat{a}_0, \ldots, \hat{a}_3)$, $\tilde{J}_4 \neq 0$. Thus, by the Implicit Function Theorem, for $(c_1, \ldots, c_8)$ sufficiently close to $(0, \ldots, 0)$, there are unique continuous functions $a_0, \ldots, a_3$ of $c_1, \ldots, c_8$ that maintain $g_1 = \cdots = g_4 = 0$.

Since every $\epsilon > 0$ neighborhood of $(c_1, \ldots, c_8) = (0, \ldots, 0)$ contains all $3^8$ possible $(+, -, 0)$ configurations, $(c_1, \ldots, c_8)$ can be chosen with appropriate signs. Also, since $(b_0, \ldots, b_3)$ is a positive vector, $(a_0, \ldots, a_3)$ is a positive vector when $(c_1, \ldots, c_8)$ is sufficiently close to $(0, \ldots, 0)$. Hence, there are positive real numbers $a_0, \ldots, a_4$ and appropriately signed $c_i$ such that (4) is satisfied.

The above proof for the case $n = 4$ relies on two facts: first, Theorem 9 guarantees the existence of a matrix $A_4 \in Q(T_4)$ with any given real characteristic polynomial; second, the Jacobian $\tilde{J}_4$ can be made arbitrarily close to $J_4 \neq 0$. Since these two statements are also true for $n = 3, 5, 6, 7$, the proof can be extended to $3 \leqslant n \leqslant 7$. $\square$

The method used to prove Theorem 14 can be used with patterns other than $T_n$. Suppose $\tilde{Y}$ is a superpattern of $Y$, and that $Y$ has been shown to be a SAP by using the method and notation of Observation 10. Let the matrix in $Q(Y)$ with positive entries $b_0, \ldots, b_k$ have the required characteristic polynomial $r(z)$, which can be taken arbitrarily close to $z^n$. Let $\tilde{X} \in Q(\tilde{Y})$ with its nonzero entries specified by the positive parameters $x_0, \ldots, x_k$ and appropriately signed $c_1, \ldots, c_t$, and set $(x_n, \ldots, x_k) = (b_n, \ldots, b_k)$. Then the Jacobian $\tilde{J}$ of the coefficients of $z^{n-1}, z^{n-2}, \ldots, z^1, z^0$ in $\det(zI - \tilde{X})$ with respect to $x_0, \ldots, x_{n-1}$ evaluated at $(x_0, \ldots, x_{n-1}, c_1, \ldots, c_t) = (b_0, \ldots, b_{n-1}, 0, \ldots, 0)$ is nonzero since $\tilde{J}$ can be made arbitrarily close to $J$ by making $(b_0, \ldots, b_{n-1})$ sufficiently close to $(\hat{x}_0, \ldots, \hat{x}_{n-1})$. Applying the Implicit Function Theorem for $(c_1, \ldots, c_t)$ sufficiently close to $(0, \ldots, 0)$, there exists $(x_0, \ldots, x_{n-1})$ arbitrarily close to $(b_0, \ldots, b_{n-1})$ such that $x_0, \ldots, x_{n-1}, b_n, \ldots, b_k, c_1, \ldots, c_t$ specify a matrix $\tilde{X} \in Q(\tilde{Y})$ with characteristic polynomial $r(z)$, thus implying that $\tilde{Y}$ is a SAP.

**Observation 15.** If $\tilde{Y}$ is a superpattern of $Y$, and $Y$ can be shown to be a SAP by the method of Observation 10, then $\tilde{Y}$ is a SAP.

This observation leads to the following conjecture.

**Conjecture 16.** If $S$ is an MSAP, then any superpattern $\tilde{S}$ is a SAP.

## 6. Further questions on SAPs

As noted in Section 1, up to equivalence $T_2$ is the only $2 \times 2$ pattern that is a SAP (IAP). For $3 \times 3$ tree patterns, in addition to $T_3$, the pattern

$$U = \begin{bmatrix} - & + & 0 \\ - & + & + \\ 0 & + & - \end{bmatrix}$$

is a MSAP (MIAP). To justify this statement, consider the matrix

$$M = \begin{bmatrix} -m_0 & 1 & 0 \\ -m_1 & m_2 & 1 \\ 0 & m_3 & -m_4 \end{bmatrix} \in Q(U).$$

If $(m_0, \ldots, m_4) = (2, 8, 3, 1, 1)$, then $M$ is nilpotent. Furthermore, the Jacobian of the coefficients of $z^2, z^1, z^0$ in $\det(zI - M)$ with respect to $m_0, m_1, m_2$ at $(2, 8, 3, 1, 1)$ is equal to $-1 \neq 0$. Thus, by Observation 10, $U$ is a SAP, and it is easily checked that it is a MSAP even though it has more than two nonzero diagonal entries.

By checking all tree patterns that have a potentially stable subpattern (see [7, Fig. 2]), we have verified (up to equivalence) that $T_3$, $U$ and the pattern $\tilde{T}_3$ obtained from $T_3$ by replacing its $(2, 2)$ zero entry by $+$ (see Theorem 14) are the only $3 \times 3$ tree patterns that are SAPs (IAPs). This observation leads to the following open questions. For $n \geqslant 4$, which $n \times n$ tree patterns are MSAPs (MIAPs, SAPs, IAPs)? By checking all $4 \times 4$ tridiagonal patterns with exactly two nonzero diagonal entries (see [7, Figs. 3, 4] and [6, Example 4.4]), apart from $T_4$ the only other MSAP (with eight nonzero entries) is the antipodal tridiagonal pattern

$$H = \begin{bmatrix} - & + & 0 & 0 \\ + & 0 & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & + & + \end{bmatrix}.$$

The fact that $H$ is a SAP is verified by using Observation 10.

By noting that the one $4 \times 4$ potentially stable tree pattern with a star graph having exactly one positive and one negative diagonal entry (see [7, Table 1]) is sign nonsingular, it follows that no such $4 \times 4$ pattern with eight nonzero entries is an IAP.

By considering all other antipodal tridiagonal $5 \times 5$ patterns (with ten nonzero entries) and using the symmetry given by Theorem 11, there is only one pattern $F$ apart from $T_5$ that has a nilpotent realization, namely

$$F = \begin{bmatrix} - & + & 0 & 0 & 0 \\ - & 0 & + & 0 & 0 \\ 0 & + & 0 & + & 0 \\ 0 & 0 & + & 0 & + \\ 0 & 0 & 0 & - & + \end{bmatrix}.$$

Since $F$ has $2n$ nonzero entries, Observation 10 can be used to verify that $F$ is an MSAP. By Observation 15, superpatterns $\tilde{U}$, $\tilde{H}$, $\tilde{F}$ are SAPs.

For $n \leqslant 7$, we have proved in Section 4 that $T_n$ is a MSAP (MIAP). For $n \geqslant 8$, we leave as another open question whether this statement is true (i.e., whether Conjectures 1 or 2 hold for all $n$). As alluded to in the Introduction, we know of no IAP that is not a SAP. Our discussion of MSAPs is focussed on antipodal tridiagonal patterns; the existence of MSAPs for other tree patterns remains open for $n \geqslant 4$.

The question of whether a general $n \times n$ sign pattern $S$ has $A \in Q(S)$ such that $A$ is nilpotent is listed as an interesting open question in [2]. For $n = 2, 3$, Theorem 4 in [9] shows (up to equivalence) that $T_2, T_3, U_3$, and $\tilde{T}_3$ are the only tree patterns with at least two nonzero diagonal entries having this property. Thus for $n = 2, 3$, the tree patterns with at least two nonzero diagonal entries that are SAPs are the same as those containing nilpotent matrices. It is unknown if this is true for $n \geqslant 4$.

As mentioned in the Introduction, in the case of $n \times n$ complex matrices, it is known [3,4,8] that many freely chosen collections of as few as $n$ entries allow spectral arbitrariness. Over the reals no such general results seem to be known, and it is clear (e.g., from considering symmetric matrices with free diagonal) that any such results must be much more restrictive. (Of course, only the eigenvalues of a real matrix could be attained.) Thus the results that we have developed here are potentially valuable prototypes in the subject of (real) inverse eigenvalue problems. To progress much further in the characterization of SAPs or to prove the natural conjecture that MSAPs are crucial (see Conjecture 16), either new general results or very good special arguments are needed.

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